

# Enhanced DOA Estimation Using Eigenvalue Reconstruction and Toeplitz Preprocessing

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**Abstract**—Reliable Direction of Arrival (DOA) estimation is crucial for the performance of wireless communication systems. In this paper, we introduce a refined DOA estimation method that combines eigenvalue reconstruction of the noise subspace and Toeplitz preprocessing with the Multiple Signal Classification (MUSIC) algorithm. The proposed technique enhances the consistency of the noise subspace and improves the algorithm's resolution. Extensive simulations demonstrate that the method outperforms both the standard MUSIC and the MUSIC with Eigenvalue Reconstruction (MUSIC\_ER) techniques. Notably, our approach shows enhanced performance in terms of root mean square error (RMSE) across snapshot ranges from 1 to 10. These enhancements make the proposed method (MUSIC\_TR) a practical and effective option, especially in low-snapshot scenarios, providing an alternative solution for DOA estimation.

**Keywords**—DOA, MUSIC, Toeplitz Preprocessing, Radar, Wireless Communication

## I. INTRODUCTION

DOA estimation is crucial in radar, wireless communications, acoustics, and medical imaging due to its significant impact on system performance and efficiency [1]. In radar systems, accurate DOA estimation enables precise target detection and tracking, essential for both civilian and military applications [2]. In wireless communications, DOA estimation optimizes signal reception, improves network capacity, and enhances user experience through advanced techniques like beamforming [1]. In acoustics, including sonar and audio processing, accurate DOA estimation is key for locating sound sources, vital for underwater navigation and advanced hearing aids [2].

Among the various techniques for DOA estimation the MUSIC algorithm is recognized for its high resolution and robustness. MUSIC estimates the angles of incoming signals by decomposing the received signal covariance matrix into signal and noise subspaces. This technique is particularly effective when there are a large number of snapshots, but its performance can degrade when only a few snapshots are available, such as in mobile or dynamic communication environments. This issue stems from the reduced accuracy in

estimating the noise subspace under low-snapshot conditions [3].

In scenarios with limited data, accurately determining the Direction of Arrival (DOA) becomes challenging due to difficulties in estimating the noise subspace. While methods like the MUSIC algorithm are highly effective in ideal conditions, their performance drops significantly when fewer snapshots are available. This decline happens because it becomes harder to clearly distinguish between the signal and noise subspaces, resulting in reduced accuracy and resolution. To tackle these issues, techniques like Toeplitz preprocessing has been explored to improve the structure of the covariance matrix, leading to more reliable noise subspace estimation. Our goal is to refine these methods further, ensuring high accuracy and resolution in low-snapshot environments by making better use of the Toeplitz matrix structure [4].

The proposed method involves reconstructing of eigenvalues in the noise subspace to minimize signal leakage and applying Toeplitz preprocessing to impose a structured form on the covariance matrix. The proposed DOA estimation method, MUSIC\_TR, can be applied in radar systems similarly by using received covariance matrix to enhance the accuracy of target detection and tracking, especially in low snapshot. Radars operating in dynamic or cluttered conditions often suffer from limited snapshot availability, making precise DOA estimation challenging. The MUSIC\_TR method's eigenvalue reconstruction and Toeplitz preprocessing can improve the uniformity of the noise subspace and enhance resolution. This would result in better target localization, even in the presence of interference or noise, making the method highly applicable for radar systems. Its robustness makes it especially useful in radar applications where accurate real-time target detection.

The rest of the paper is structured as follows. Section 2 provides an overview of the background and related work. In Section 3, we explain the methodology used for the proposed framework. The results are discussed in Section 4. Lastly, Section 5 concludes the paper and suggests directions for future research.

## II. BACKGROUND AND RELATED WORK

The MUSIC algorithm was introduced by Schmidt in 1986. It remains one of the most popular subspace-based methods for DOA estimation [5]. MUSIC exploits the perpendicularity between signal and noise subspaces through eigenvalue decomposition of the covariance matrix of received signals. This allows for high resolution DOA estimation but suffers significant performance degradation under low Snapshots conditions. Accurate covariance matrix estimation is crucial, especially with finite snapshots and strong interference [2].

Adaptive beamforming techniques, such as Capon's method, enhance signal reception by dynamically adjusting the beam pattern based on the received signal environment [6]. These methods minimize interference and noise while maximizing the desired signal. However, they require precise knowledge of the array geometry. Additionally, they need an accurate understanding of the signal environment. This may not always be feasible. Model mismatches and finite sample effects can adversely affect performance, necessitating robust adaptations like diagonal loading [7].

The Minimal Resource Allocation Network (MRAN) is a neural network-based approach for DOA estimation that adapts its structure based on input data. MRAN performs well in non-stationary environments and outperforms traditional methods like MUSIC, especially in noisy conditions. Despite these advantages, MRAN's effectiveness heavily depends on selecting the optimal network size, which impacts performance. Additionally, its practical deployment is challenged by computational complexity and the need for extensive training data [1]. Noise subspace-based methods, such as the Noise Subspace Projection (NSP) algorithm proposed by Ali Olfat and Said Nader-Esfahan, utilize the invariance of noise subspace eigenvalues to improve DOA estimation accuracy [8]. However, like other subspace-based techniques, NSP is limited by finite snapshot issues and inaccuracies in noise covariance matrix estimation. Recent advancements have employed more robust statistical techniques and adaptive processing methods to overcome these limitations [9].

Despite advancements, several limitations persist in DOA estimation techniques. Subspace-based methods like MUSIC and NSP are highly sensitive to noise and interference, requiring precise covariance matrix estimation, which is challenging with finite snapshots. Neural network-based methods like MRAN demand extensive training data and computational resources, making them less practical for real-time applications. Recent research has focused on improving the robustness and accuracy of covariance matrix estimation to address the limitations of traditional DOA methods. Shrinkage-based techniques have emerged as effective solutions for dealing with finite sample sizes and improving covariance matrix estimation accuracy. These methods balance bias and variance by combining sample covariance matrices with structured estimators, leading to more reliable DOA estimates. Additionally, advancements in machine learning and statistical signal processing have introduced new algorithms that adaptively reconstruct the noise subspace, further improving DOA estimation performance under challenging conditions [10].

Moreover, in DOA estimation, the Ziv-Zakai Bound has become a more effective tool than the Cramér-Rao Bound, especially in non-asymptotic cases. The ZZB provides a tighter bound on mean square error across a wider range of signal-to-noise ratios. This makes it better for predicting performance in complex scenarios with multiple sources. The ZZB is also effective in handling permutation ambiguities, whether the sources are coherent or incoherent. For these reasons, the ZZB is essential in improving DOA estimation methods and should be a key consideration in performance evaluation [11].

While traditional DOA estimation methods like MUSIC, adaptive beam formers, and MRAN have significantly advanced the field, each has inherent limitations. Recent innovations in covariance matrix estimation and adaptive processing offer promising avenues to enhance the accuracy of DOA estimation. These advancements pave the way for more reliable and efficient algorithms.

## III. METHODOLOGY

This section explains the methodology used for DOA estimation, covering the formulation of the signal model, eigenvalue reconstruction in the noise subspace, Toeplitz preprocessing, and how these techniques are integrated into the MUSIC algorithm. Each step is thoroughly described with mathematical details to show how these enhancements improve accuracy in the estimation process.

### A. Signal Model and Problem Formulation

Consider a Uniform Linear Array (ULA) with  $M$  sensors receiving  $K$  narrowband signals from different directions. The incoming signal at the sensor can be modeled as:

$$y(t) = A(\theta)s(t) + n(t) \quad (1)$$

In eq.no (1)  $y(t)$  is  $M \times 1$  received signal vector,  $A(\theta)$  is the  $M \times K$  steering matrix with  $\theta$  representing the angles of arrival,  $s(t)$  is the  $K \times 1$  source signal vector, and  $n(t)$  is the  $M \times 1$  noise vector supposed to be zero-mean Gaussian noise. The covariance matrix of the received signals is given by:

$$R_y = A(\theta)R_{ss}A^H(\theta) + \sigma^2 I \quad (2)$$

In eq.no (2)  $R_{ss} = E[s(t)s^H(t)]$  is the source covariance matrix and  $\sigma^2 I$  is the noise covariance matrix. The goal is to estimate the DOAs  $\theta$  from the observed data  $y(t)$ .

In real time application, it is hard to achieve ideal covariance matrix as shown in eq. (2). The quality of the covariance matrix relies on the number of snapshots. The sample covariance matrix  $\tilde{R}_y$  is computed from the received signal snapshots [2]:

$$\tilde{R}_y = \frac{1}{T} \sum_{t=1}^T y(t)y^H(t) \quad (3)$$

In eq.no (3)  $T$  is the number of snapshots and  $y^H(t)$  is the Hermitian or complex conjugate transpose of the signal vector  $y(t)$ . This matrix represents the averaged outer product of the signal vectors over  $T$  snapshots offering an estimated covariance matrix of the signal.

To enhance the resolution and improve estimation accuracy, the eigenvalue reconstruction of the noise subspace can be helpful [12]. The signal and noise subspaces are separated using this technique by taking advantage of the characteristics of the covariance matrix of the received signal. The process involves the following steps:

- Apply eigenvalue decomposition (EVD) on the sample covariance matrix  $\tilde{R}_y$  [5]:

$$\tilde{R}_y = V\Lambda V^H \quad (4)$$

where  $\Lambda$  is a diagonal matrix containing the eigenvalues, and the accompanying matrix consists of the corresponding eigenvectors.

- While the eigenvectors corresponding to the remaining  $M-K$  eigenvalues encompass the noise domain, the eigenvectors corresponding to the  $K$  greatest eigenvalues cover the signal subspace. The noise eigenvalues are averaged to form a reconstructed noise eigenvalue [13]. Identify the  $K$  largest eigenvalues corresponding to the signal subspace and reconstruct the remaining  $M-K$  noise eigenvalues by averaging [12]:

$$\hat{\lambda}_{noise} = \frac{1}{M-K} \sum_{i=K+1}^M \lambda_i \quad (5)$$

For  $i = K+1, \dots, M$ , this results in a modified eigenvalue of matrix  $\Lambda'$  and  $\lambda_i$  is noise eigenvalues of sample covariance matrix.

- The reconstructed covariance matrix is then formed using the original signal eigenvalues and the reconstructed noise eigenvalue [12]:

$$\tilde{R}'_y = V_s \Lambda_s V_s^H + \hat{\lambda}_{noise} V_n V_n^H \quad (6)$$

Where  $\Lambda_s$  and  $V_s$  are the eigenvalues and eigenvectors of the signal subspace and  $V_n$  is the matrix of eigenvectors of the noise subspace and  $\hat{\lambda}_{noise}$  reconstructed noise eigenvalues.

### B. Toeplitz Preprocessing

Toeplitz preprocessing helps enhance the covariance matrix by imposing a structured form, making it more suitable for DOA estimation. Here are the steps involved [4]:

- Formation of Toeplitz Matrix: Construct a Toeplitz matrix  $R_{Toeplitz}$  by averaging the elements along each diagonal of the reconstructed covariance matrix  $R'_y$  [5]:
- Ensuring Positive Semi-Definiteness: Ensure the resulting Toeplitz matrix is positive semi-definite, which can involve additional eigenvalue adjustments if necessary [2].

Toeplitz preprocessing helps regularize the covariance matrix, thereby improving its estimation accuracy.

After Toeplitz preprocessing to  $R'_y$  to obtain  $R_{Toeplitz}$ :

$$R_{Toeplitz}(i, j) = \frac{1}{N-|i-j|} \sum_{k=0}^{N-|i-j|-1} R'_y(i+k, j+k) \quad (7)$$

In eq.no (7)  $N$  is the size of the matrix and  $|i-j|$  is the absolute difference between the row and column indices.  $K$  is index used in summation. Here we apply the Toeplitz preprocessing on the Reconstructed Eigen values in noise subspace.

To estimate the DOAs, the MUSIC pseudo spectrum is computed using the noise subspace eigenvectors and the steering vector for various angles. The pseudo spectrum  $P_{MUSIC}(\theta)$  Produces a plot where the peaks indicate the DOAs.

$$P_{MUSIC}(\theta) = \frac{1}{a^H(\theta)E_n E_n^H a(\theta)} \quad (8)$$

where  $a(\theta)$  is the steering vector and  $E_n$  contains the eigenvectors of noise subspace.

The precision of the DOA algorithm is evaluated by calculating the RMSE. This measure is defined as follows:

$$RMSE = \sqrt{\frac{1}{NK} \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} |\hat{\theta}_{k,c} - \theta_k|} \quad (9)$$

The error is computed as the square root of the mean of the squared differences between the predicted values  $\hat{\theta}_{k,c}$  the actual values  $\theta_k$ . Equations (1)-(9) is maintained as describe the mathematical formulation used to estimate the covariance matrix apply eigenvalue decomposition and reconstruct the noise subspace all of which are integral to DOA estimation. Equation (9) shows how the root mean square error (RMSE) is computed to evaluate the DOA estimation accuracy. RMSE is an essential performance metric that directly reflects the error in estimating the DOA. A lower RMSE indicates higher accuracy in DOA estimation, as shown in the simulations where the MUSIC\_TR method consistently achieved lower RMSE values. However, RMSE is not directly proportional to DOA but is a measure of the estimation accuracy. The smaller the RMSE, the closer the estimated DOAs are to the true DOAs, hence the relationship between RMSE and the precision of the DOA estimation process.

## IV. RESULTS

### A. Performance Metrics RMSE Vs. SNR

To evaluate the performance of the proposed MUSIC\_TR method, we conducted simulations using a uniform linear array (ULA) with  $M = 20$  sensors, each spaced at half the wavelength. The array received a narrowband signal from a single source  $K = 1$  with an angle of arrival fixed at  $\theta = 10.1^\circ$ . The evaluation focused on comparing the RMSE against SNR for different snapshot values.

In the first test as illustrate in Figure 1, we set the number of snapshots  $T=1$  representing a minimal snapshot scenario. The results show improvement of the proposed MUSIC\_TR algorithm over both the traditional MUSIC algorithm and the MUSIC\_ER method. At such a low snapshot, DOA estimation can be particularly challenging as we can see the improvements occurs after SNR -10db. This indicates that the proposed method is more effective in low-snapshot conditions.

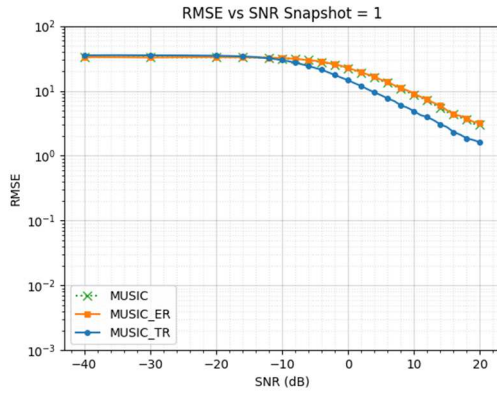


Figure 1. Single Source RMSE vs SNR

Next, we increased the number of snapshots to  $T=3$  as illustrating in Figure 2 while maintaining the same angle. Even with just three snapshots, the proposed MUSIC\_TR method continues to perform better than the traditional MUSIC algorithm. Although the gap between MUSIC\_TR and MUSIC\_ER narrows slightly, MUSIC\_TR still offers a notable improvement in RMSE, showing that the reconstruction of eigenvalues and the Toeplitz preprocessing help stabilize performance with limited data.

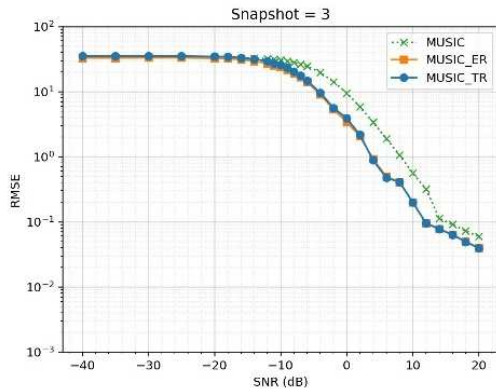


Figure 2. Single Source 3 Snapshots RMSE vs SNR

Finally, we evaluated the performance of the algorithms with  $T=10$  snapshots as shown in Figure 3. In this case, the results indicate that MUSIC\_TR performs comparably to both MUSIC and MUSIC\_ER. This suggests that, as the number of snapshots increases, the benefits of the proposed method become less pronounced. These results indicate that the proposed MUSIC\_TR method is particularly effective for low-snapshot scenarios, where traditional methods struggle due to inaccurate covariance matrix estimation. As the number of snapshots increases, the performance advantage of MUSIC\_TR becomes less significant.

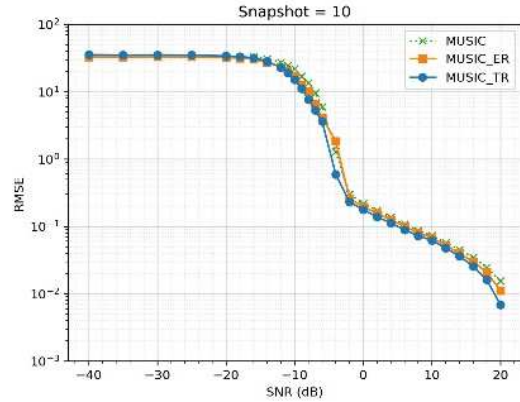


Figure 3. Single Source 10 Snapshots RMSE vs SNR

### B. Pseudo Spectrum Analysis

In addition to RMSE analysis, we conducted a pseudo-spectrum comparison to visually assess the resolution and accuracy of the proposed MUSIC\_TR algorithm. The pseudo-spectrum plots provide an intuitive way to evaluate the ability of each method to correctly estimate the direction of arrival. Simulations were conducted using a uniform linear array ULA with same setting as above but the SNR is fixed at 10db. The number of snapshots  $T$  was varied to examine its effect on estimation accuracy.

The pseudo spectrum comparison for 1 snapshot, as shown in the Figure 4, provides a clear contrast between the three methods: MUSIC, MUSIC\_ER, and the proposed MUSIC\_TR. The traditional MUSIC algorithm displays multiple small peaks across a range of angles, which may lead to less accurate DOA estimations. MUSIC\_ER also exhibits several false peaks, indicating some limitations in precision. In comparison, the MUSIC\_TR offers improved resolution and more reliable DOA estimation in one-snapshot conditions, demonstrating its ability to maintain accuracy even with limited data.

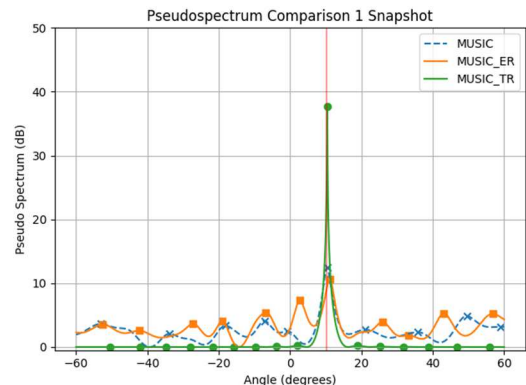


Figure 4. Pseudo spectra Comparison Snapshot 1

For  $T=3$  snapshots as shown in Figure 5, the traditional MUSIC algorithm continues to exhibit several small peaks, which could affect the accuracy of DOA estimation. MUSIC\_ER shows improvement by getting closer to MUSIC\_TR by reducing the number of side peaks. On other

hand the proposed MUSIC\_TR proposed method still performing well with almost no side peaks.

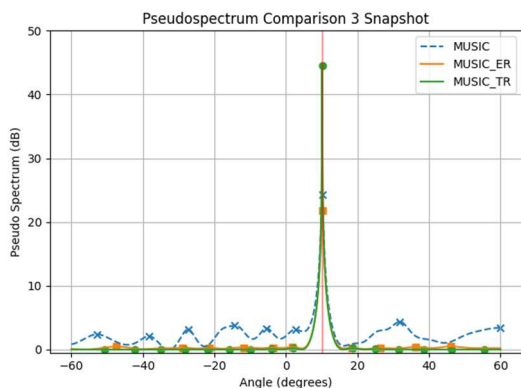


Figure 5. Pseudo spectra Comparison Snapshot 3

For the case of 10 snapshots as shown in Figure 6, we observe almost same performance from all three methods. The traditional MUSIC algorithm shows improvement compared to the lower snapshot scenarios, with fewer side lobes, indicating an increase in resolution as more snapshots are utilized. However, the proposed MUSIC\_TR method maintains its performance, delivering a clean pseudo-spectrum with minimal false peaks. Despite the convergence in performance across all methods with more snapshots, this result demonstrates that while the proposed method excels in low-snapshot conditions.

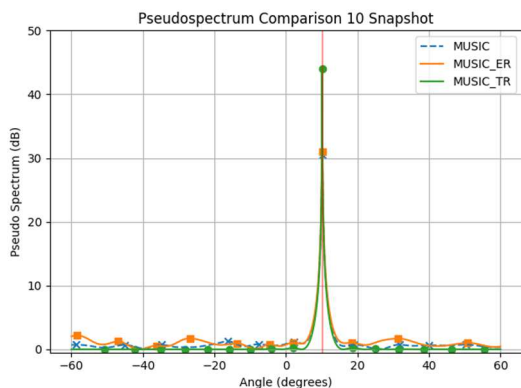


Figure 6. Pseudo spectra Comparison Snapshot 10.

## V. CONCLUSION

In this paper, we proposed an enhanced DOA estimation method by combining eigenvalue reconstruction in the noise subspace with Toeplitz preprocessing, applied within the framework of the MUSIC algorithm. The proposed method showcases improvements in DOA estimation accuracy over traditional MUSIC and MUSIC\_ER. Our method performing well in low snapshots conditions it especially it's performing well on minimum possible one snapshot. The results indicate that, as snapshots increases the performance of MUSIC\_TR converges with that of MUSIC\_ER and MUSIC. This suggests that in scenarios with a higher number of snapshots, the benefits of the proposed method diminish. In the future, we can enhance it more with the integration of Machine learning some advance techniques like to use deep neural networks to automatically learn the optimal parameters for covariance matrix estimation.

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